

Invariants of the spherical sector in conformal mechanics

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A direct relation is established between the constants of motion for conformal mechanics and those for its spherical part. In this way we find the complete set of functionally independent constants of motion for the so-called cuboctahedric Higgs oscillator, which is just the spherical part of the rational A_3 Calogero model (describing four Calogero particles after decoupling their center of mass).

I. INTRODUCTION

Recently, there has been new interest in so-called “conformal mechanics”. This term denotes a system whose Hamiltonian H , together with the dilatation generator D and the generator K of conformal boosts forms, with respect to Poisson brackets, the conformal algebra $so(1,2)$:

$$\{H, D\} = 2H, \quad \{K, D\} = -2K, \quad \{H, K\} = D. \quad (1)$$

Such system can always be presented in the form [1]

$$D = p_r r, \quad K = \frac{r^2}{2}, \quad H = \frac{p_r^2}{2} + \frac{\mathcal{I}(u)}{2r^2}, \quad (2)$$

where the radial coordinates (r, p_r) and the angular coordinates (u^α) obey the basic Poisson brackets

$$\{p_r, r\} = 1, \quad \{u^\alpha, p_r\} = \{u^\alpha, r\} = 0, \quad \{u^\alpha, u^\beta\} = (\omega^{-1})^{\alpha\beta}(u). \quad (3)$$

The spherical (or angular) part of the Hamiltonian H ,

$$\mathcal{I} = 4KH - D^2, \quad (4)$$

is the Casimir element of (1) and, hence, commutes with all generators but also defines a constant of motion of the Hamiltonian H .

The spherical part of the conformal mechanics, determined by

$$\omega_0 := \frac{1}{2}\omega_{\alpha\beta}du^\alpha \wedge du^\beta \quad \text{and} \quad \mathcal{I}, \quad (5)$$

may be considered as a Hamiltonian system by itself. We refer to it as “spherical mechanics” throughout the paper. It is obvious that integrability of the initial conformal mechanics leads to integrability of the “spherical mechanics” (ω_0, \mathcal{I}) , and vice versa. It is also evident that the constants of motion of the spherical mechanics are constants of motion for the conformal mechanics. Yet, the inverse is generally not true, although there should be a way to construct the “spherical” constants of motion out of the “conformal” ones. This is the problem we address in this paper.

In [1] some of us began a study of spherical mechanics. It is relevant for investigations of the Calogero model [2, 3] and its various extensions and generalizations [4] (for a recent review see [5]). Furthermore, the spherical mechanics of the rational A_N Calogero model defines the multi-center (Higgs) oscillator system on the $N-1$ -sphere [6]. The

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well-known series of hidden constants of motion found by Wojciechowski [7] for the Calogero model has a transparent explanation in terms of spherical mechanics, and its analog exists in any integrable conformal mechanical system [1]. Even in the simplest case of $N=2$, the one-dimensional spherical mechanics of the A_2 Calogero model shed light on a global aspect of Calogero models, by elucidating the non-equivalence of different quantizations of the Calogero model [8]. The $\mathcal{N}=4$ superconformal generalizations of the rational A_2 Calogero model, constructed via supersymmetrization of spherical mechanics [9], yielded a scheme for lifting any $\mathcal{N}=4$ supersymmetric mechanics to a $D(1, 2|\alpha)$ superconformal one [1]. Finally, a formulation in terms of action-angle variables [11] led to the equivalence of the rational A_2 and G_2 Calogero models and provided restrictions on the “decoupling” transformation which maps the Calogero model to the free-particle system considered in [12, 13].

Directly relevant for the task of the present paper, it was in fact demonstrated in [1] that all information on a conformal mechanics system is encoded in its spherical part. In particular, the “conformal” constants of motion with even conformal dimension were shown to induce constants of motion for (ω_0, \mathcal{I}) . However, the authors were unable to find the “spherical” constants of motion induced by the odd-dimensional initial constants of motion. In the following, we are going to solve this problem with the help of $so(3)$ representation theory.

The paper is arranged as follows: In Section II, following but extending [1], we relate the symmetries of conformal mechanics to the particular system of differential equations on the spherical phase space. The analysis is simplified by the use of $so(3)$ representations, which clarifies the origin of the spin operators appearing in the final system. In the Section III we construct a series of the constants of motion for the spherical mechanics, which is induced by the constants of motion (of any conformal dimension) for the conformal system. In Section IV we apply our method to the rational A_3 Calogero model and derive the complete set of functionally independent constants of motion for the cuboctahedric Higgs oscillator.

II. THE SPHERICAL PART OF CONFORMAL MECHANICS (“SPHERICAL MECHANICS”)

In this section, we relate the constants of motion of the conformal mechanics (2) with certain differential equations on the phase space of the associated spherical mechanics. The result of this section appeared already in [1], but the current formulation is given in terms of $so(3)$ representations.

For any function f on phase space, define the associated Hamiltonian vector field by the Poisson bracket action $\hat{f} = \{f, \cdot\}$. For example, the Hamiltonian vector fields corresponding to the generators H, D, K (2), and Casimir element (4) read

$$\hat{H} = p_r \frac{\partial}{\partial r} + \frac{\mathcal{I}}{r^3} \frac{\partial}{\partial p_r} + \frac{\hat{\mathcal{I}}}{2r^2}, \quad \hat{K} = -r \frac{\partial}{\partial p_r}, \quad \hat{D} = r \frac{\partial}{\partial r} - p_r \frac{\partial}{\partial p_r}, \quad (6)$$

$$\text{and} \quad \hat{\mathcal{I}} = 4H\hat{K} + 4K\hat{H} - 2D\hat{D}. \quad (7)$$

Since the assignment $f \mapsto \hat{f}$ is a Lie algebra homomorphism, the vector fields $\hat{H}, \hat{K}, \hat{D}$ satisfy the $so(1, 2)$ algebra (1), and the vector field of the Casimir element $\hat{\mathcal{I}}$, of course, commutes with them.

Any constant of motion is the lowest weight vector of the conformal algebra (1), since it is annihilated by the Hamiltonian. Without any restriction, one can choose it to have a certain conformal dimension (spin):

$$\hat{H}I_s = 0, \quad \hat{D}I_s = -2sI_s. \quad (8)$$

A conformal mechanics which describes identical particles and possesses a permutation-invariant cubic (in momenta, $s=3/2$) constant of motion commuting with the total momentum yields the rational Calogero model, which is an integrable system [18].

In the following, we consider only nonnegative integer and half-integer values of the spin s , so that I_s yields a finite-dimensional (nonunitary) representation of the $so(1, 2)$ algebra (6). This includes the N -particle rational Calogero model and its extensions, whose Liouville constants of motion are polynomials in the momenta.

Our goal is to derive the constants of motion for the “spherical” Hamiltonian (4) from the constants of motion of the initial conformal Hamiltonian. Using (2), (6), and (7) it is easy to see that the conservation condition (8) is equivalent to the equation

$$(\hat{\mathcal{I}} - \hat{M}) I_s(p_r, r, u) = 0, \quad \text{where} \quad \hat{M} = 2(\hat{S}_- - \mathcal{I}\hat{S}_+). \quad (9)$$

Here, the one-dimensional vector fields \hat{S}_\pm together with \hat{S}_z are given by

$$\hat{S}_+ = \frac{1}{r} \frac{\partial}{\partial p_r}, \quad \hat{S}_- = -p_r r^2 \frac{\partial}{\partial r}, \quad \hat{S}_z = -\frac{1}{2} \left(r \frac{\partial}{\partial r} + p_r \frac{\partial}{\partial p_r} \right). \quad (10)$$

Interestingly, they form an $so(3)$ algebra,

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z, \quad [\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm. \quad (11)$$

Note that \hat{S}_+ is generated by the Hamiltonian $S_+ = -\log(r)$ while the other two vector fields are not Hamiltonian.

The integral (8) can be presented as a sum of terms with decoupled radial and angular coordinates and momenta [19],

$$I_s(p_r, r, u) = \sum_{m=-s}^s f_{s,m}(u) R_{s,m}(p_r, r) \quad \text{with} \quad R_{s,m}(p_r, r) = \sqrt{\binom{2s}{s+m}} \frac{p_r^{s-m}}{r^{s+m}}. \quad (12)$$

The radial functions $R_{s,m}$ form a spin s -representation ($s = 0, \frac{1}{2}, \dots$) of the $so(3)$ algebra (11),

$$\hat{S}_+ R_{s,m} = \sqrt{(s-m)(s+m+1)} R_{s,m+1}, \quad \hat{S}_- R_{s,m} = \sqrt{(s-m+1)(s+m)} R_{s,m-1}, \quad \hat{S}_z R_{s,m} = m R_{s,m}. \quad (13)$$

Hence, $\hat{\mathcal{I}}$ acts nontrivially only on the angular functions, while the \hat{S}_a act on the radial ones. Due to the convolution (12), one can shift the latter action to the angular functions by transposing the $so(3)$ matrices. As a result, the action of $\hat{\mathcal{I}}$ on the spin- s states $f_{s,m}$ is given by

$$\hat{\mathcal{I}} f_{s,m} = \sum_{m'} M_{mm'} f_{s,m'} = 2(\sqrt{(s-m)(s+m+1)} f_{s,m+1} - \mathcal{I} \sqrt{(s-m+1)(s+m)} f_{s,m-1}). \quad (14)$$

This is a system of $2s+1$ first-order linear homogeneous differential equations for the angular functions $f_{s,m}(u)$. The coefficients depend only on \mathcal{I} , which commutes with the differential operator, and so they can be treated as constants. Note that all angular coefficients must obey the related $(2s+1)$ th-order linear homogeneous differential equation

$$\text{Det}(\hat{\mathcal{I}} - M) f_{s,m} = 0, \quad (15)$$

which is, in fact, equivalent to the system (14), since any solution f of (15) also generates a solution of the original system. Indeed, using (14), one can recursively express each function $f_{s,m}$ as a $(s \pm m)$ th-order polynomial in $\hat{\mathcal{I}}$ acting on the function $f_{s,\mp s}$. Diagonalization of the matrix M decouples the system (14) into independent equations, pertaining to the eigenvalues and eigenvectors of the vector field $\hat{\mathcal{I}}$.

Consider now some consequences of the relation (14). From a constant of motion of the Hamiltonian, one can construct other constants with the same conformal spin by successive application of the vector field generated by the spherical Hamiltonian:

$$I_s \xrightarrow{\hat{\mathcal{I}}} I_s^{(1)} \xrightarrow{\hat{\mathcal{I}}} I_s^{(2)} \xrightarrow{\hat{\mathcal{I}}} \dots \xrightarrow{\hat{\mathcal{I}}} I_s^{(k)} \xrightarrow{\hat{\mathcal{I}}} \dots, \quad I_s^{(k)} := \hat{\mathcal{I}}^k I_s. \quad (16)$$

In general, the members of this sequence are not in involution. At most the first $2s+1$ integrals can be independent, while the remaining ones are expressed through them linearly with \mathcal{I} -dependent coefficients, since the vector field $\hat{\mathcal{I}}$ acts on the $(2s+1)$ -vector of constants $I_s^{(k)}$ as a square matrix with \mathcal{I} -valued entries. The exact amount of functionally independent integrals depends on the I_s as well as on the concrete realization of the conformal mechanics.

III. CONSTANTS OF MOTION OF THE SPHERICAL MECHANICS

In this Section we present the construction of the constants of motion for the spherical mechanics (ω_0, \mathcal{I}) from those for the initial conformal mechanics, based on $so(3)$ group representations. This method yields constants of motion of any conformal dimension and recovers the expressions found in [1].

Any constant of motion I_s of the original Hamiltonian is given by its coefficients in the decomposition (12). The related conservation condition (9), (14), or (15) is decoupled into independent equations upon diagonalization of the matrix M ,

$$\hat{M} = 4\sqrt{-\mathcal{I}} \hat{U} \hat{S}_z \hat{U}^{-1}, \quad \text{where} \quad \hat{U} = (-\mathcal{I})^{\frac{1}{2}\hat{S}_z} e^{-\frac{i\pi}{2}\hat{S}_y} \quad \text{with} \quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-). \quad (17)$$

Thus, up to an \mathcal{I} -valued factor, the vector field \hat{M} is equivalent to the usual spin- z projection operator. The operator $\exp(-\frac{i\pi}{2}\hat{S}_y)$ maps \hat{S}_z to \hat{S}_x . The latter is then transformed to \hat{M} by the operator $(-I)^{\hat{S}_z/2}$, which, for the present, means a formal power series. Together with the factor $i\sqrt{\mathcal{I}}$ it contains square roots of \mathcal{I} . Thus \hat{M} is, in general, complex and multi-valued. When the potential is positive, as is the case in Calogero models, the spherical part is strictly positive, and the operator (17) is complex but single-valued. In any case, all square roots will cancel in the final expressions for the constants of motion.

Define now the rotated basis for the algebra (11), which is formed by the eigenstates of the operator \hat{M} . Using (17), we obtain

$$\begin{aligned} \tilde{R}_{s,m} &= (\hat{U}R)_{s,m} = \sum_{m'} U_{m'm} R_{s,m'}, \quad U_{m'm} = d_{m'm}^s(\pi/2)(-\mathcal{I})^{\frac{m'}{2}}, \\ \hat{M}\tilde{R}_{s,m} &= m\tilde{R}_{s,m}, \end{aligned} \quad (18)$$

where $d_{m'm}^s(\beta)$ is the Wigner's small d -matrix, which describes the rotation around the y axis in the usual spin- s representation (13). The explicit expressions are given in the Appendix, see (A1), (??), (A4). There we have also collected some formulae and relations among the d -matrix elements which are relevant for this article. Note that the functions $\tilde{R}_{s,m}$ now depend on the angular variables also through \mathcal{I} . The integral (12) of the original Hamiltonian can be presented in terms of the rotated functions as

$$I_s(p_r, r, u) = \sum_{m=-s}^s \tilde{f}_{s,m}(u) \tilde{R}_{s,m}(p_r, r, \mathcal{I}(u)). \quad (19)$$

The new coefficients are expressed in terms of old ones by means of the inverse transformation [compare (12) with (19) and (18)]:

$$\tilde{f}_{s,m} = \sum_{m'} U_{mm'}^{-1} f_{s,m'} = \sum_{m'} (-\mathcal{I})^{-\frac{m'}{2}} d_{m'm}^s(\pi/2) f_{s,m'}. \quad (20)$$

In the second equation, we have applied the orthogonality condition of the d -matrix [the first equation in (A3)]. Substituting the decomposition (19) into (9) and using the eigenvalue-eigenfunction equation form (19), we arrive at a similar eigensystem equation for the vector field $\hat{\mathcal{I}}$ and the rotated angular coefficients:

$$\hat{\mathcal{I}}\tilde{f}_{s,m}(u) = 4m\sqrt{-\mathcal{I}(u)}\tilde{f}_{s,m}(u). \quad (21)$$

This provides a formal solution to the system (14). For systems with positive spherical part, the eigenvalue is a well-defined purely imaginary function, and the evolution of the coefficients driven by the spherical Hamiltonian oscillate with a frequency proportional to m ,

$$\tilde{f}_{s,m}(t) = e^{i\omega_m(t-t_0)} \tilde{f}_{s,m}(t_0) \quad \text{with} \quad \omega_m = 4m\sqrt{\mathcal{I}}. \quad (22)$$

Various combinations of these quantities give rise to constants of motion for the spherical Hamiltonian. In particular, for integer spin s , the zero-frequency coefficient $\tilde{f}_{s,0}(u)$ is an integral itself. Using the explicit expression of the Wigner d -matrix for this case (A5), one can express it in terms of the original coefficients:

$$\begin{aligned} \mathcal{J}_s(u) &= \mathcal{I}(u)^{\frac{s}{2}} \tilde{f}_{s,0}(u) = \sum_{m=-s}^s \frac{(s+m-1)!!(s-m-1)!!}{\sqrt{(2s)!}} \delta_{s-m,2\mathbb{Z}} \mathcal{I}(u)^{\frac{s-m}{2}} f_{s,m}(u) \\ &= \sum_{\ell=0}^s \frac{(2\ell-1)!!(2s-2\ell-1)!!}{\sqrt{(2s)!}} \mathcal{I}(u)^\ell f_{s,2\ell-s}(u). \end{aligned} \quad (23)$$

Here, \mathbb{Z} denotes the set of integer numbers, so that $\delta_{k,2\mathbb{Z}} = 1$ for even values of k and vanishes for the odd values. The supplementary \mathcal{I} -dependent factor in front of the angular coefficient eliminates the fractional powers of \mathcal{I} ,

leaving only integral powers of \mathcal{I} in the final expression. Up to a normalization factor, (23) coincides with the expression (5.2) of [1].

For integer values of s , the same integral can also be obtained from the equivalent higher-order differential equation (15). Indeed, due to (17) or (21), the related differential operator takes the following form:

$$\text{Det}(\hat{\mathcal{I}} - M) = \prod_{m=-s}^s (\hat{\mathcal{I}} - 4m\sqrt{-\mathcal{I}}) =: \begin{cases} \hat{\mathcal{I}}\hat{\Delta}_s & \text{for } s \in \mathbb{Z}, \\ \hat{\Delta}_s & \text{for } s \in \mathbb{Z} + \frac{1}{2}, \end{cases} \quad \text{with } \hat{\Delta}_s = \prod_{0 < m \leq s} (\hat{\mathcal{I}}^2 + 16m^2\mathcal{I}). \quad (24)$$

Therefore, for integer spin value, (15) is reduced to

$$\hat{\mathcal{I}}\hat{\Delta}_s f_{s,m} \equiv \prod_{m=1}^s (M^2 + 16m^2\mathcal{I}) f_{s,m} = 0, \quad (25)$$

which implies that $\hat{\Delta}_s f_{s,m}$ is an integral of motion of \mathcal{I} . The operator $\hat{\Delta}_s$ projects out all but one of the eigenfunctions $\tilde{f}_{s,m}$,

$$\hat{\Delta}_s \tilde{f}_{s,m} = \delta_{m0}(s!)^2 (16\mathcal{I})^s \tilde{f}_{s,m}. \quad (26)$$

Therefore, the above integral has to be proportional to (23). This can be verified independently if we apply $\hat{\Delta}_s$ to both sides of the inversion of (20) and use (18), (23), (A5):

$$\hat{\Delta}_s f_{s,m} = U_{m0} \hat{\Delta}_s \tilde{f}_{s,0} = \delta_{s-m,2\mathbb{Z}} c_{s,m} \mathcal{I}^{\frac{s+m}{2}} \mathcal{J}_s \quad \text{with } c_{s,m} = (-8i)^s s! \binom{s}{\frac{s+m}{2}} \sqrt{(s-m)!(s+m)!}. \quad (27)$$

How to construct an integral of \mathcal{I} from an integral of H with half-integral conformal spin? The corresponding representation has no $m=0$ state, but one can consider such a state in the integral $I_{2s} = I_s^2$, which has integral spin value equal to $2s$. In general, integrals of \mathcal{I} can be built also by bilinear combinations of $f_{s,m}(u)$ with opposite values of the spin projection. In fact, any state

$$\begin{aligned} \mathcal{J}_s^m &= (-\mathcal{I})^s \tilde{f}_{s,m} \tilde{f}_{s,-m} = \sum_{m',m''} i^{4s+m''-m'} d_{m''m}^s(\pi/2) d_{m'm}^s(\pi/2) \mathcal{I}^{s-\frac{m'+m''}{2}} f_{s,m'} f_{s,m''} \\ &= \sum_{m',m''} \delta_{m''-m',2\mathbb{Z}} (-1)^{2s+\frac{m''-m'}{2}} d_{m''m}^s(\pi/2) d_{m'm}^s(\pi/2) \mathcal{I}^{s-\frac{m'+m''}{2}} f_{s,m'} f_{s,m''} \end{aligned} \quad (28)$$

is an integral of \mathcal{I} . In the first equation, we have used the symmetry property (A6) of the d -matrix. The Kronecker delta appears after symmetrization over the two summation indices in the first double sum, with the help of

$$\frac{1}{2}(i^{m''-m'} + i^{m'-m''}) = i^{m''-m'} \frac{1}{2}(1 + (-1)^{m'-m''}) = i^{m''-m'} \delta_{m''-m',2\mathbb{Z}}. \quad (29)$$

Therefore, the constant of motion \mathcal{J}_s^m of the spherical Hamiltonian is a real polynomial of order $2s$ in \mathcal{I} .

There is a clear interpretation of the constructed integrals in terms of representation theory. Take some set of angular functions satisfying (9) or (14), which means that the related quantity I_s (12) is an integral of H . Then, according to the tensor product of $so(3)$ representations, one can construct other sets of angular functions,

$$f_{S,m}(u) = \sum_{m_1+m_2=m} C_{s,m_1,s,m_2}^{S,m} f_{s,m_1}(u) f_{s,m_2}(u) \quad \text{with } S = 2s, 2s-2, \dots, \quad (30)$$

which satisfy a similar equation. The multiplets with odd values of $S-2s$ are absent in the symmetric tensor product, due to the exchange symmetry of the Clebsch-Gordan coefficients (B3). From the angular functions (30) one can compose “new” integrals of the original Hamiltonian via

$$I'_S = \sum_m f_{S,m} R_{S,m} \quad \text{with } S = 2s, 2s-2, \dots, \quad (31)$$

each corresponding to a symmetric multiplet in the tensor product of two spin- s multiplets. Note that the first integral from this set just coincides with the square of the original integral, $I'_{2s} = I_s^2$, as can easily be verified using

(B2). Since S is always integer, the related multiplet contains an $m = 0$ state, which is a constant of motion of the spherical Hamiltonian:

$$\mathcal{F}_s^S(u) = \sum_m C_{s,m,s,-m}^{S,0} \mathcal{J}_m^s(u). \quad (32)$$

Unwanted fractional powers of \mathcal{I} cancel as before. These two sets of integrals are, of course, equivalent.

A similar “blending” procedure can be applied to the mixing of two different integrals I_{s_1} and I_{s_2} with integer value of $s_1 - s_2$. The resulting integrals of \mathcal{I} are parameterized by the whole set of $2s_{\min} + 1$ different angular momenta obeying the sum rule.

The construction straightforwardly generalizes also to multilinear forms composed from the angular functions. The expression (28) expands to

$$\mathcal{J}_{s_1 \dots s_k}^{m_1 \dots m_k}(u) = \mathcal{I}(u)^{\frac{1}{2} \sum_{\ell} s_{\ell}} \prod_{\ell=1}^k \tilde{f}_{s_{\ell}, m_{\ell}}(u) \quad \text{with} \quad \sum_{\ell=1}^k m_{\ell} = 0, \quad (33)$$

where the last relation implies that the total spin $\sum s_{\ell}$ must be an integer. These observables can be combined into a single multiplet of integer spin S by a $(k-1)$ -fold application of the Clebsch-Gordan decomposition. The final set of observables $\tilde{f}_{S,m}$ forms an integral of the original Hamiltonian, while its $m=0$ element corresponds to an integral of the spherical Hamiltonian.

So far, we have only considered products of the angular functions. More generally however, one could also employ fractions of them, with the same spin projection of the numerator and the denominator, such as $\tilde{f}_{s_1,m}/\tilde{f}_{s_2,m}$. Of course, this entails introducing singularities, which might create problems for the quantization due to inverse powers of moments.

It has to be mentioned that the variety of angular constants of motion constructed here are not independent. It may even happen that some of them vanish. Moreover, the compatibility of the integrals of motion for H does not at all yet imply the compatibility of the associated integrals for \mathcal{I} , as can be seen from (28).

Examples

At the end of this section, we demonstrate our method by presenting some simplest examples for the obtained constants of motion.

First we note that there exist two bilinear conserved quantities (28) and (32), which have a rather simple form in terms of the original angular coefficients. The first one is the canonical “singlet”, which is the same both in the original and the rotated basis,

$$\mathcal{F}_s^0(u) \sim \sum_m (-1)^{s-m} \tilde{f}_{s,m} \tilde{f}_{s,-m} = \sum_m (-1)^{s-m} f_{s,m} f_{s,-m}. \quad (34)$$

The second one is given by the trivial superpositions of the states (28), which is reduced by the orthogonality of the d -matrices to

$$\sum_m \mathcal{J}_s^m \sim \sum_m \mathcal{I}^{s-m} f_{s,m}^2. \quad (35)$$

For the integral I_s of the Hamiltonian H with conformal spin $s = \frac{1}{2}$, the general formula (28) takes its simplest form, up to a normalization factor,

$$\mathcal{J}_{\frac{1}{2}}^{\frac{1}{2}} \sim \mathcal{I} f_{\frac{1}{2}, -\frac{1}{2}}^2 + f_{\frac{1}{2}, \frac{1}{2}}^2. \quad (36)$$

Consider now the integral with conformal spin $s=1$ of the original Hamiltonian. The related linear integral of \mathcal{I} is (see (23))

$$\mathcal{J}_1 \sim \mathcal{I} f_{1,1} + f_{1,-1}. \quad (37)$$

In addition, there are two quadratic integrals given by (28), one of which ($\mathcal{J}_{s=1}^{m=0}$) is the square of the above integral, while the other one can be identified with either (34) or (35). The Hamiltonian itself can be considered as a particular case. For $I_1 = H$, the coefficient f_{10} vanishes while the others become constants, so the sole constant of \mathcal{I} extracted from H is \mathcal{I} itself.

The first nontrivial case corresponds to the next conformal spin $s = \frac{3}{2}$, when there is no linear but two independent quadratic integrals. The simplest choice then are the two functions (34) and (35).

IV. FOUR-PARTICLE CALOGERO MODEL

Let us use the general method developed in the previous section to construct the complete set of constants of motion for the spherical mechanics of the four-particle Calogero model after decoupling the center of mass (i.e. of the rational A_3 Calogero model). This spherical mechanics also describes a particle on the two-dimensional sphere, interacting by the Higgs-oscillator law with force centers located in the vertices of a cuboctahedron. By this reason, the system was termed “cuboctahedric Higgs oscillator” [6].

We remind that the standard rational Calogero model,

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i < j} \frac{g^2}{(x_i - x_j)^2}, \quad (38)$$

has N Liouville constants of motion, given in terms of a Lax matrix by the expression [5]

$$I_s = \text{Tr } L^{2s} \quad \text{with} \quad s = \frac{1}{2}, 1, \dots, \frac{N}{2}, \quad (39)$$

where

$$L_{jk} = \delta_{jk} p_k + (1 - \delta_{jk}) \frac{ig}{x_j - x_k}. \quad (40)$$

Hence, $I_{\frac{1}{2}} = \sum_i p_i$ and $I_1 = H$. Furthermore, the quantities

$$I_s^{(1)} = \hat{L} I_s \quad \text{for} \quad s \neq 1 \quad (41)$$

coincide with Wojciechowski’s additional integrals [1]. Together with (39), they form a complete set of functionally independent integrals making the system maximally superintegrable [7].

We choose $N=4$ and pass to new coordinates

$$y_0 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4), \quad y_1 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4), \quad y_2 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4), \quad y_3 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4) \quad (42)$$

and associated momenta \tilde{p}_i with $i = 0, 1, 2, 3$. This transformation decouples the center-of-mass coordinate y_0 and momentum \tilde{p}_0 from the others. After setting

$$y_0 = \tilde{p}_0 = 0, \quad (43)$$

the Hamiltonian takes the form of the rational $D_3 \sim A_3$ Calogero model [6]

$$H = \frac{1}{2} \sum_{i=1}^3 \tilde{p}_i^2 + \sum_{i,j=1}^3 \left(\frac{g^2}{(y_i - y_j)^2} + \frac{g^2}{(y_i + y_j)^2} \right) = \frac{1}{2} p_r^2 + \frac{\mathcal{I}(p_\theta, p_\varphi, \theta, \varphi)}{2r^2}. \quad (44)$$

In the second equation, we introduced spherical coordinates (r, θ, φ) on $\mathbb{R}^3(y_1, y_2, y_3)$ together with their conjugate momenta $(p_r, p_\theta, p_\varphi)$, so that

$$\mathcal{I}(p_\theta, p_\varphi, \theta, \varphi) = p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} + \frac{2g^2}{\sin^2 \theta} \sum_{\pm} \left[\frac{1}{(\cos \varphi \pm \sin \varphi)^2} + \frac{1}{(\cot \theta \pm \sin \varphi)^2} + \frac{1}{(\cot \theta \pm \cos \varphi)^2} \right], \quad (45)$$

in accord with the spherical symplectic structure $\omega_0 = dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi$.

According to (39) and (40), the conformal Hamiltonian (44) has two Liouville constants of motion of conformal dimension three and four, given by

$$I_{\frac{3}{2}} = \text{Tr}(L^3) = \sum_{i=1}^4 p_i^3 + \dots = 3\tilde{p}_1\tilde{p}_2\tilde{p}_3 + \dots = \frac{3}{2}p_r^3 \cos \theta \sin^2 \theta \sin 2\varphi + \dots, \quad (46)$$

$$\begin{aligned} I_2 = \text{Tr}(L^4) &= \sum_{i=1}^4 p_i^4 + \dots = \frac{1}{4}(\tilde{p}_1^4 + \tilde{p}_2^4 + \tilde{p}_3^4) + \frac{3}{2}(\tilde{p}_1^2\tilde{p}_1^2 + \tilde{p}_1^2\tilde{p}_3^2 + \tilde{p}_2^2\tilde{p}_3^2) + \dots \\ &= \frac{1}{4}p_r^4 (\sin^2 2\theta + \sin^4 \theta \sin^2 2\varphi + 1) + \dots \end{aligned} \quad (47)$$

Here, we have written out only the terms of highest order in the radial momentum. Comparing (46) and (47) with (12), we obtain the spherical functions $f_{s,-s}$ as the coefficients of the monomials p_r^{2s} ,

$$f_{\frac{3}{2}}(\theta, \varphi) = \frac{3}{2} \cos \theta \sin^2 \theta \sin 2\varphi, \quad f_2(\theta, \varphi) = \frac{1}{4} (\sin^2 2\theta + \sin^4 \theta \sin^2 2\varphi). \quad (48)$$

Here and in the following, we use for convenience the shorter notation

$$f_s(\theta, \varphi) := f_{s,-s}(\theta, \varphi). \quad (49)$$

The Liouville integrals (46) and (47) are supplemented by the two related Wojciechowski integrals $I_{\frac{3}{2}}^{(1)}$ and $I_2^{(1)}$ (41), whose leading-term coefficients are (see (41))

$$g_{\frac{3}{2}} = \hat{\mathcal{I}} f_{\frac{3}{2}} \quad \text{and} \quad g_2 = \hat{\mathcal{I}} f_2. \quad (50)$$

Note that the f_s depend on the angles only while the g_s are linear in the angular momenta. Together with the Hamiltonian (44), we obtain a complete set $\{H, I_{\frac{3}{2}}, I_{\frac{3}{2}}^{(1)}, I_2, I_2^{(1)}\}$ of five independent integrals.

In order to derive the Poisson algebra of integrals, we compute first the commutators between the related coefficients:

$$\begin{aligned} \{f_{\frac{3}{2}}, g_{\frac{3}{2}}\} &= 18(f_{\frac{3}{2}}^2 - f_2), & \{f_2, g_2\} &= 8(4f_2^2 - \frac{1}{3}f_{\frac{3}{2}}^2 - f_2), & \{f_{\frac{3}{2}}, g_2\} &= \{f_2, g_{\frac{3}{2}}\} = 8f_{\frac{3}{2}}(3f_2 - 1), \\ \{f_{\frac{3}{2}}, f_2\} &= 0, & \{g_{\frac{3}{2}}, g_2\} &= 4(2g_{\frac{3}{2}}f_2 - 3f_{\frac{3}{2}}g_2). \end{aligned} \quad (51)$$

Since the map $I_s \rightarrow f_s$ is a Poisson algebra homomorphism [1], we immediately get the analogous relations for the constants of motion by inserting powers of $2H$ in order to balance the conformal spins on both sides of the equations (the coefficient for the Hamiltonian (44) is a constant: $f_1 = \frac{1}{2}$). Thus, the nontrivial Poisson brackets are

$$\begin{aligned} \{I_{\frac{3}{2}}, I_{\frac{3}{2}}^{(1)}\} &= 18(I_{\frac{3}{2}}^2 - 2I_2H), & \{I_2, I_2^{(1)}\} &= 8(4I_2^2 - \frac{2}{3}I_{\frac{3}{2}}^2H - 4I_2H^2), \\ \{I_{\frac{3}{2}}, I_2^{(1)}\} &= \{I_2, I_{\frac{3}{2}}^{(1)}\} = 8I_{\frac{3}{2}}(3I_2 - 4H^2), & \{I_{\frac{3}{2}}^{(1)}, I_2^{(1)}\} &= 4(2I_{\frac{3}{2}}^{(1)}I_2 - 3I_{\frac{3}{2}}I_2^{(1)}). \end{aligned} \quad (52)$$

This is a particular realization of part of the quadratic algebra related to the Hamiltonian [16] (see [17] for rational Calogero models based on arbitrary root systems). It is expressed in terms of independent generators, therefore higher orders appear on the right-hand sides.

We now derive a complete set of functionally independent constants of motion for the spherical mechanics of the four-particle Calogero model. The second expression in (24) immediately yields the spherical constant of motion associated with (47),

$$\mathcal{J}_2 = -\frac{1}{\sqrt{6}} \left(\frac{1}{256} \hat{\mathcal{I}}^4 + \frac{5}{16} \mathcal{I} \hat{\mathcal{I}}^2 + 4\mathcal{I}^2 \right) f_2. \quad (53)$$

Its explicit expression, which can be calculated using (45) and (48), is highly complicated,

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{\sqrt{6}} \left[\frac{1}{16} (3 \cos 4\varphi - 11) p_\theta^4 - \frac{3}{4} \cot \theta \sin 4\varphi p_\theta^3 p_\varphi - \left(\frac{11+9 \cos 4\varphi}{8 \sin^2 \theta} + \frac{9}{4} \sin^2 2\varphi \right) p_\theta^2 p_\varphi^2 \right. \\ &\quad \left. + \frac{3}{4} \cot^3 \theta \sin 4\varphi p_\theta p_\varphi^3 + \frac{3 \cos^4 \theta \cos 4\varphi + 21 \sin^4 \theta - 18 \sin^2 \theta - 11}{16 \sin^4 \theta} p_\varphi^4 \right] \\ &\quad + g^2 K_1(\theta, \varphi) p_\theta^2 + g^2 K_2(\theta, \varphi) p_\theta p_\varphi + g^2 K_3(\theta, \varphi) p_\varphi^2 + g^4 K_4(\theta, \varphi), \end{aligned} \quad (54)$$

where the functions $K_1(\theta, \varphi), K_2(\theta, \varphi), K_3(\theta, \varphi), K_4(\theta, \varphi)$ are given in Appendix C.

The system of equations (14) can be applied in order to express the coefficients $f_{\frac{3}{2},m}$ in terms of the “lowest” one:

$$f_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{2\sqrt{3}} \hat{\mathcal{I}} f_{\frac{3}{2}}, \quad f_{\frac{3}{2},\frac{1}{2}} = \left(\frac{1}{8\sqrt{3}} \hat{\mathcal{I}}^2 + \frac{\sqrt{3}}{2} \mathcal{I} \right) f_{\frac{3}{2}}, \quad f_{\frac{3}{2},\frac{3}{2}} = \left(\frac{1}{48} \hat{\mathcal{I}}^2 + \frac{7}{12} \mathcal{I} \right) \hat{\mathcal{I}} f_{\frac{3}{2}}. \quad (55)$$

Then, using (45) and (48), one obtains the spherical constants of motion (28) associated with (46), namely $\mathcal{J}_{\frac{3}{2}}$ and $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$. Their explicit expressions are rather lengthy:

$$\begin{aligned} \mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}} = & -\frac{3}{32} \sin^2 2\varphi p_\theta^6 - \frac{3}{16} \cot \theta \sin 4\varphi p_\theta^5 p_\varphi - \frac{3}{128 \sin^2 \theta} (6 \cos^2 \theta + (13 - 3 \cos 2\theta) \cos 4\varphi) p_\theta^4 p_\varphi^2 + \frac{3}{2} \cot \theta \sin 4\varphi p_\theta^3 p_\varphi^3 \\ & - \frac{3}{128 \sin^4 \theta} (22 \sin^4 \theta - (43 - 53 \cos 2\theta) \cos 4\varphi \cos^2 \theta + 6 \cos 2\theta) p_\theta^2 p_\varphi^4 - \frac{3}{32 \sin^5 \theta} (7 - 9 \cos 2\theta) \cos^3 \theta \sin 4\varphi p_\theta p_\varphi^5 \\ & - \frac{3 \cos^2 \theta}{128 \sin^6 \theta} ((5 + 11 \cos 4\varphi) \sin^2 \theta + (2 - 9 \cos 2\theta \sin^2 \theta)(1 - \cos 4\varphi)) p_\varphi^6 + \text{terms of lower order in } p_\theta \text{ and } p_\varphi, \end{aligned} \quad (56)$$

$$\begin{aligned} \mathcal{J}_{\frac{3}{2}} = & -\frac{9}{32} \sin^2 2\varphi p_\theta^6 - \frac{9}{16} \cot \theta \sin 4\varphi p_\theta^5 p_\varphi - \frac{9}{64} \left(\frac{5 \cos 4\varphi + 3}{\sin^2 \theta} + 10 \sin^2 2\varphi \right) p_\theta^4 p_\varphi^2 \\ & - \frac{9}{64 \sin^4 \theta} (5 \cos^4 \theta \cos 4\varphi + 10 \sin^2 \theta - 5 \sin^4 \theta + 3) p_\theta^2 p_\varphi^4 + \frac{9}{16} \cot^5 \theta \sin 4\varphi p_\theta p_\varphi^5 \\ & + \frac{9 \cos^2 \theta}{64 \sin^6 \theta} (\cos^4 \theta \cos 4\varphi - 6 \sin^2 \theta - \sin^4 \theta - 1) p_\varphi^6 + \text{terms of lower order in } p_\theta \text{ and } p_\varphi. \end{aligned} \quad (57)$$

Clearly, \mathcal{I} , \mathcal{J}_2 , $\mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}}$ and $\mathcal{J}_{\frac{3}{2}}$ cannot be functionally independent, since our spherical mechanics has a four-dimensional phase space. Indeed, using **Mathematica**, we uncover the following algebraic relation,

$$\mathcal{J}_{\frac{3}{2}}^{\frac{3}{2}} = \frac{1}{3} \mathcal{J}_{\frac{3}{2}}^{\frac{1}{2}} + 2\sqrt{\frac{2}{3}} \mathcal{J}_2 \mathcal{I} + \frac{1}{3} \mathcal{I}^3 + 4g^2 \mathcal{I}^2. \quad (58)$$

This is the only relation among the four constants of motion, since (56) and (57) are not in involution with (54). Even their free-particle parts ($g=0$ projects to the terms of highest order in the momenta) do not commute as is easy to verify. Hence, we have found three functionally independent spherical constants of motion for the A_3 Calogero model. This confirms the superintegrability of that system.

V. CONCLUSION

In the present paper we have developed a general approach to the constants of motion for conformal mechanics, based on $so(3)$ representation theory. In particular, we gave an explicit construction of the (overcomplete set of) constants of motion for the spherical part of conformal mechanics (“spherical mechanics”), which are related to the constants of motion for the initial conformal system. We have illustrated the effectiveness of our method on the example of the rational A_3 Calogero model and its spherical mechanics (which defines the cuboctahedric Higgs oscillator). For the latter we have constructed a complete set of functionally independent constants of motion, proving its intuitively obvious superintegrability.

Unfortunately, our approach does not allow one to select a commuting subset of constants of motion for the spherical mechanics. Also, it does not provide us with a rule for selecting *a priori* functionally independent constants of motions. Hopefully, further development of this approach will provide answers to these questions.

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Appendix A: Wigner (small) d -matrix

The spin- j representation of the rotation group parameterized by three Euler angles is given by the Wigner D -matrix [14, 15]. We only need the (small) d -matrix, which describes the rotation around the y axis,

$$d_{m'm}^s(\beta) = \langle sm' | \exp(-i\beta S_y) | sm \rangle, \quad (A1)$$

where $m, m' = -s, \dots, s$ are the spin z -projection quantum numbers. Its elements are real and given by [14]

$$d_{m'm}^s(\beta) = \sum_t (-1)^{t+m'-m} \frac{\sqrt{(s+m')!(s-m')!(s+m)!(s-m)!}}{(j+m-t)!(m'-m+t)!(j-m'-t)!} \left(\cos \frac{\beta}{2} \right)^{2s+m-m'-2t} \left(\sin \frac{\beta}{2} \right)^{m'-m+2t}, \quad (A2)$$

where the sum is over such values of t that the factorials in the denominator are nonnegative. The elements obey

$$d_{m'm}^s(\beta) = d_{mm'}^s(-\beta) = (-1)^{m-m'} d_{mm'}^s(\beta) = d_{-m-m'}^s(\beta). \quad (A3)$$

For $\beta = \pi/2$, the above expression simplifies to

$$d_{m'm}^s(\pi/2) = 2^{-s} \sum_t (-1)^{t+m'-m} \frac{\sqrt{(s+m')!(s-m')!(s+m)!(s-m)!}}{(s+m-t)!(m'-m+t)!(s-m'-t)!}. \quad (\text{A4})$$

Further simplifications occur when one of the spin-projection quantum numbers vanishes, which is possible for integer spins only:

$$\begin{aligned} d_{m0}^s(\pi/2) &= (-1)^{\frac{s+m}{2}} \delta_{s-m,2\mathbb{Z}} \frac{\sqrt{(s-m)!(s+m)!}}{2^s \left(\frac{s+m}{2}\right)! \left(\frac{s-m}{2}\right)!} = (-1)^{\frac{s+m}{2}} \delta_{s-m,2\mathbb{Z}} \sqrt{\frac{(s+m-1)!!(s-m-1)!!}{(s+m)!!(s-m)!!}} \\ d_{0m}^s(\pi/2) &= (-1)^{\frac{s-m}{2}} \delta_{s-m,2\mathbb{Z}} \frac{\sqrt{(s-m)!(s+m)!}}{2^s \left(\frac{s+m}{2}\right)! \left(\frac{s-m}{2}\right)!} = (-1)^{\frac{s-m}{2}} \delta_{s-m,2\mathbb{Z}} \sqrt{\frac{(s+m-1)!!(s-m-1)!!}{(s+m)!!(s-m)!!}} \\ d_{00}^s(\pi/2) &= (-1)^{\frac{s}{2}} \delta_{s,2\mathbb{Z}} \frac{(s-1)!!}{s!} \end{aligned} \quad (\text{A5})$$

The factor $\delta_{s-m,2\mathbb{Z}}$ excludes odd values of $s-m$, for which the matrix elements vanish. For $\beta = \pi/2$, the relations are supplemented by

$$d_{m'm}^s(\pi/2) = (-1)^{s+m'} d_{m'-m}^s(\pi/2) = (-1)^{s-m} d_{-m'm}^s(\pi/2), \quad (\text{A6})$$

which can be obtained from $d_{m'm}^s(\pi) = (-1)^{s-m} \delta_{m',-m}$.

Appendix B: Clebsch-Gordan Coefficients

Clebsch-Gordan coefficients are the expansion coefficients of total-spin eigenstates $|SM\rangle$ in terms of the product basis $|s_1 m_1 s_2 m_2\rangle$ of eigenstates of the two coupled spins,

$$C_{s_1, m_1, s_2, m_2}^{S, M} = \langle s_1 m_1 s_2 m_2 | SM \rangle. \quad (\text{B1})$$

The general expression is complicated, but special cases are often quite simple like for the highest total-spin value:

$$C_{s_1, m_1, s_2, m_2}^{s_1+s_2, m_1+m_2} = \sqrt{\frac{\binom{2s_1}{s_1-m_1} \binom{2s_2}{s_2-m_2}}{\binom{2s_1+2s_2}{s_1+s_2-m_1-m_2}}}. \quad (\text{B2})$$

The Clebsch-Gordan coefficients have an even-odd exchange symmetry depending on the total-spin value,

$$C_{s_1, m_1, s_2, m_2}^{s, m} = (-1)^{s_1+s_2-s} C_{s_2, m_2, s_1, m_1}^{s, m}. \quad (\text{B3})$$

Appendix C: Coefficients of \mathcal{J}_2

Below we write down the explicit expressions for the functions K_1, K_2, K_3, K_4 , which appear in (54).

$$\begin{aligned} K_1(\theta, \varphi) &= \frac{1}{2^{14} \sqrt{6} \cos^2 2\varphi (\cos^2 \theta - \sin^2 \theta \cos^2 \varphi)^2 (2 \sin^2 \theta \cos 2\varphi + 3 \cos 2\theta + 1)} \times \\ &\quad \left(768 (25 + 29 \cos 2\theta) \sin^6 \theta \cos 12\varphi + 96 (1370 + 2327 \cos 2\theta + 1542 \cos 4\theta + 393 \cos 6\theta) \sin^2 \theta \cos 8\varphi \right. \\ &\quad \left. - (119258 + 175774 \cos 2\theta + 45096 \cos 4\theta + 57723 \cos 6\theta - 10242 \cos 8\theta + 5607 \cos 10\theta) \sin^{-2} \theta \cos 4\varphi \right. \\ &\quad \left. + (1021064 + 365088 \cos 2\theta - 223008 \cos 4\theta - 183840 \cos 6\theta - 61800 \cos 8\theta - 655360 \sin^{-2} \theta) \right), \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} K_2(\theta, \varphi) &= \frac{3 \cot \theta \tan 2\varphi}{8 \sqrt{6} \sin^2 \theta (17 \cos 4\theta + 28 \cos 2\theta - 8 \sin^4 \theta \cos 4\varphi + 19)^2} \times \\ &\quad \left(351 \cos 10\theta + 1350 \cos 8\theta + 13779 \cos 6\theta + 9992 \cos 4\theta + 35022 \cos 2\theta - 13824 \sin^8 \theta \cos^2 \theta \cos 8\varphi + 5042 \right. \\ &\quad \left. - 64 (81 \cos 6\theta + 702 \cos 4\theta + 1071 \cos 2\theta + 962) \sin^4 \theta \cos 4\varphi \right), \end{aligned} \quad (\text{C2})$$

$$K_3(\theta, \varphi) = \frac{1}{16\sqrt{6} \cos^2 2\varphi (17 \cos 4\theta + 28 \cos 2\theta - 8 \sin^4 \theta \cos 4\varphi + 19)^2} \times \quad (C3)$$

$$\begin{aligned} & \left(162 (13 \sin 2\varphi + \sin 6\varphi)^2 \cos 8\theta + 24 (3898 - 1569 \cos 4\varphi - 282 \cos 8\varphi + \cos 12\varphi) \cos 6\theta \right. \\ & + 36 (6686 + 1931 \cos 4\varphi - 430 \cos 8\varphi + 5 \cos 12\varphi) \cos 4\theta + 72 (546 + 10587 \cos 4\varphi - 898 \cos 8\varphi + 5 \cos 12\varphi) \cos 2\theta \\ & - (1087746 - 1625907 \cos 4\varphi + 46158 \cos 8\varphi + 483 \cos 12\varphi) \\ & \left. + 262144 (5 - 4 \cos 4\varphi) \sin^{-2} \theta - 32768 (11 - 3 \cos 4\varphi) \sin^{-4} \theta \right), \end{aligned}$$

$$K_4(\theta, \varphi) = \frac{-1}{64\sqrt{6} ((60 \cos 2\theta + 33 \cos 4\theta + 35) \cos 2\varphi - 8 \sin^4 \theta \cos 6\varphi)^4} \times \quad (C4)$$

$$\begin{aligned} & \left[64(335698872 \cos 2\theta + 204278376 \cos 4\theta + 100740648 \cos 6\theta + 30799596 \cos 8\theta + 3629304 \cos 10\theta \right. \\ & + 515160 \cos 12\theta - 649944 \cos 14\theta - 194643 \cos 16\theta + 197597863) \cos 8\varphi \\ & + 384 \sin^4 \theta ((-16777208 \cos 2\theta - 15290507 \cos 4\theta - 10272396 \cos 6\theta - 4824234 \cos 8\theta - 2019708 \cos 10\theta \\ & - 312741 \cos 12\theta - 8174886) \cos 12\varphi - 768 \sin^8 \theta (828 \cos 2\theta + 243 \cos 4\theta + 617) \cos 2\varphi \\ & - 32 \sin^4 \theta (290832 \cos 2\theta + 188916 \cos 4\theta + 81648 \cos 6\theta + 13851 \cos 8\theta + 166129) \cos 16\varphi) \\ & + \sin^{-4} \theta ((-9941103400 \cos 2\theta + 11541549238 \cos 4\theta + 10411072176 \cos 6\theta + 8259070392 \cos 8\theta \\ & + 4658511600 \cos 10\theta + 1965778311 \cos 12\theta + 569460204 \cos 14\theta + 67528026 \cos 16\theta - 29495988 \cos 18\theta \\ & - 8028477 \cos 20\theta + 4163058670) \cos 4\varphi + 62158979032 \cos 2\theta + 46026533130 \cos 4\theta + 27521060688 \cos 6\theta \\ & + 12943186248 \cos 8\theta + 4533912336 \cos 10\theta + 1033949913 \cos 12\theta - 11388780 \cos 14\theta - 94673178 \cos 16\theta \\ & \left. - 31001292 \cos 18\theta - 6738147 \cos 20\theta + 34904741074) \right]. \end{aligned}$$

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